# Understanding randomness with polynomials

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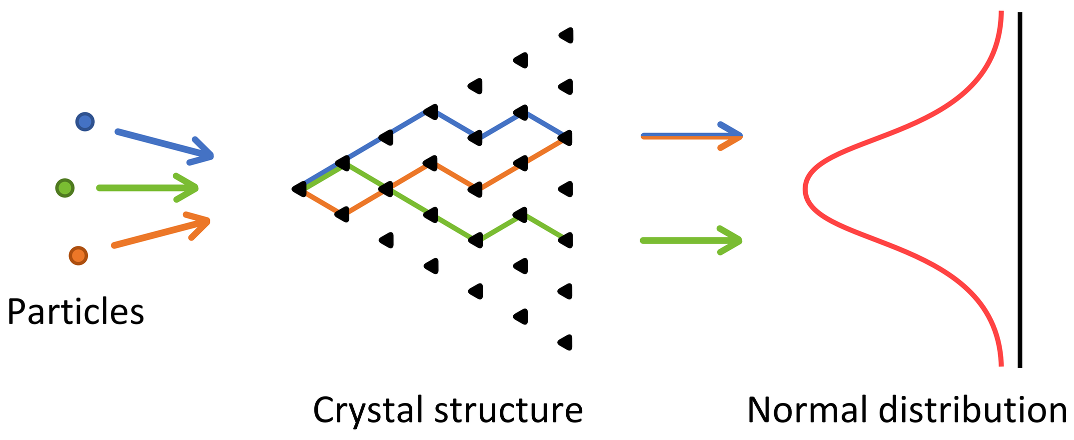
## Introduction

Many real-world phenomena evolve in time but experience fluctuations or randomness:

- The movement of small particles in a fluid
- Genetic mutations (alleles) in a population
- Market prices for stocks and commodities
- The activity of neurons in the human brain



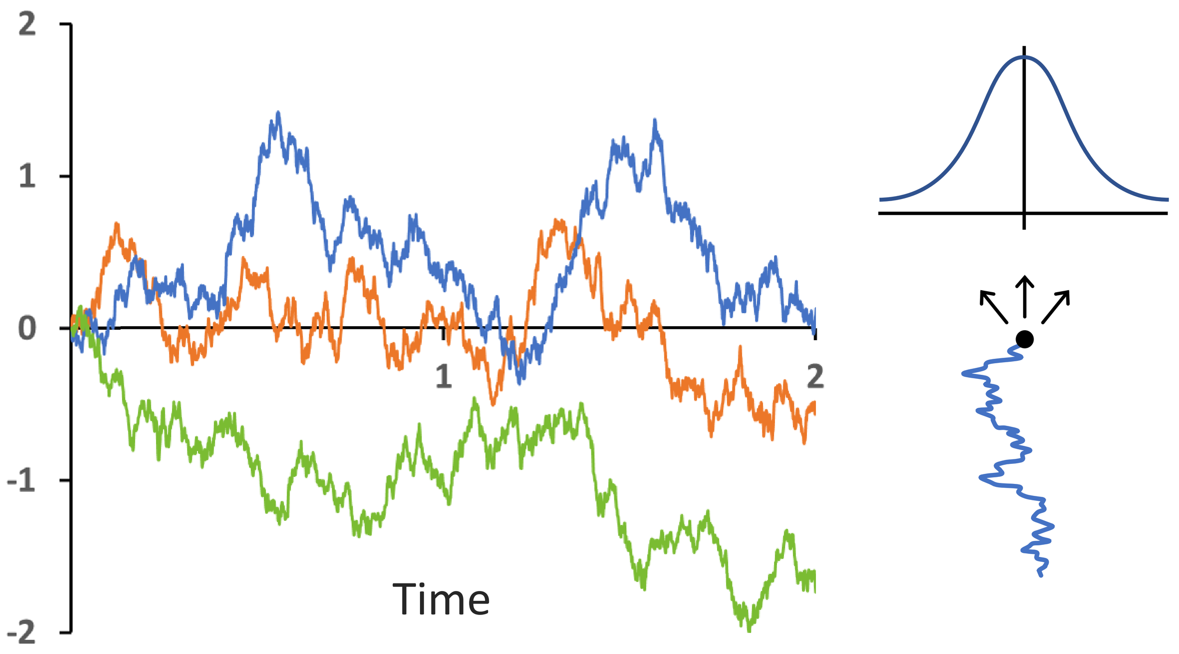

In mathematics, the areas of **Probability and Stochastic Analysis** provide the tools from which we model such random systems.

## Brownian motion

One of the key concepts is **Brownian motion** which models the random movements of a point particle with the **normal distribution**.

Brownian motion is named after the botanist Robert Brown who observed pollen grains in water exhibiting small random fluctuations.

The theory of Brownian motion was later developed by Albert Einstein in one of his famous *Annus Mirabilis* papers of 1905.




Mathematically speaking, Brownian motion is the unique stochastic process $W$ such that

1. $W = \{W_t\}_{t\geq 0}$ evolves continuously in time.
2. $W$ has random fluctuations which satisfy
$$W_t - W_s \sim \mathcal{N}\big(0, t-s\big),$$
for $s \leq t$ ($\mathcal{N}$ is the normal distribution).
3. $W$ has independent fluctuations. That is, $(W_t - W_s)$ and $(W_v - W_u)$ are independent for $u \leq v \leq s \leq t$.

## Stochastic modelling with Brownian motion

Brownian motion is a simple model that (on its own) cannot describe complex systems.

Therefore, it is often used as a building block to create more powerful **stochastic models**.

One approach is to view Brownian motion as the source of randomness within a system.

This type of mathematical model is known as the **stochastic differential equation (SDE):**

$$dy_t = \mu(t, y_t)\,dt + \sigma(t, y_t)\,dW_t,$$

where $y_t$ is the state of the system at time $t$ and the functions $\mu$ and $\sigma$ govern the small deterministic and random changes of $y$.

SDEs have widespread applications in STEM.

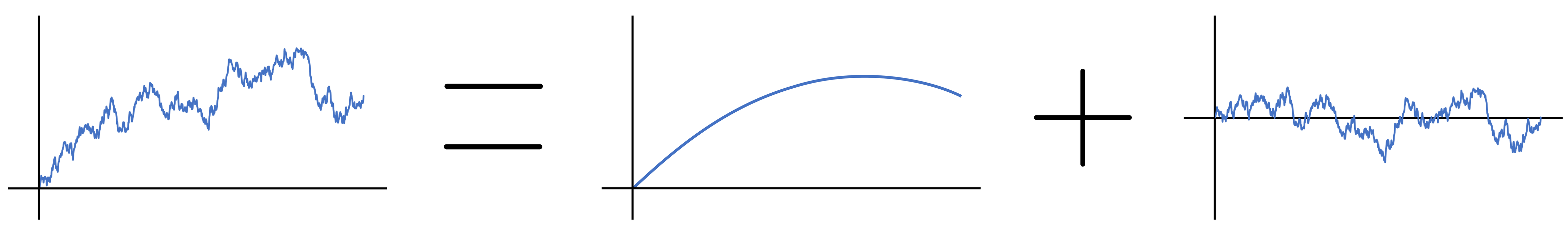
## Brownian motion and random polynomials

Our research is based on a recent discovery that connects Brownian motion with another part of classical mathematics: **polynomials**.

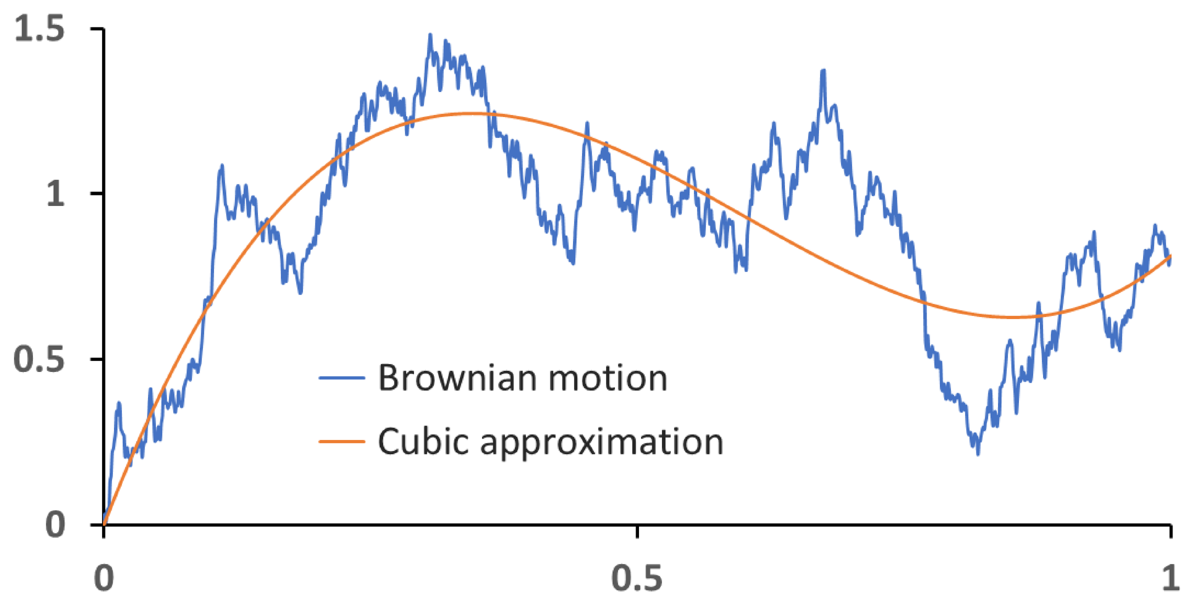
**Theorem 1 (Brownian motion is a random polynomial with some independent noise** Foster et al. (2020) and Habermann (2020)**).** *For $n \geq 1$, we have*

$$W = W^n + Z^n,$$

*where $W^n$ is a polynomial with degree $n$ and $Z^n$ is an independent process defined on $[0,1]$ with mean zero $\mathbb{E}[Z^n] = 0$ and $Z_0^n = Z_1^n = 0$.*



Since Theorem 1 is well known when $n = 1$, we have focused on the quadratic and cubic polynomial decompositions of $W$ $(n = 2, 3)$.




## Application to SDE simulation

**Newton's second law** models a particle that moves in a field with potential $f : \mathbb{R}^d \to \mathbb{R}$ as

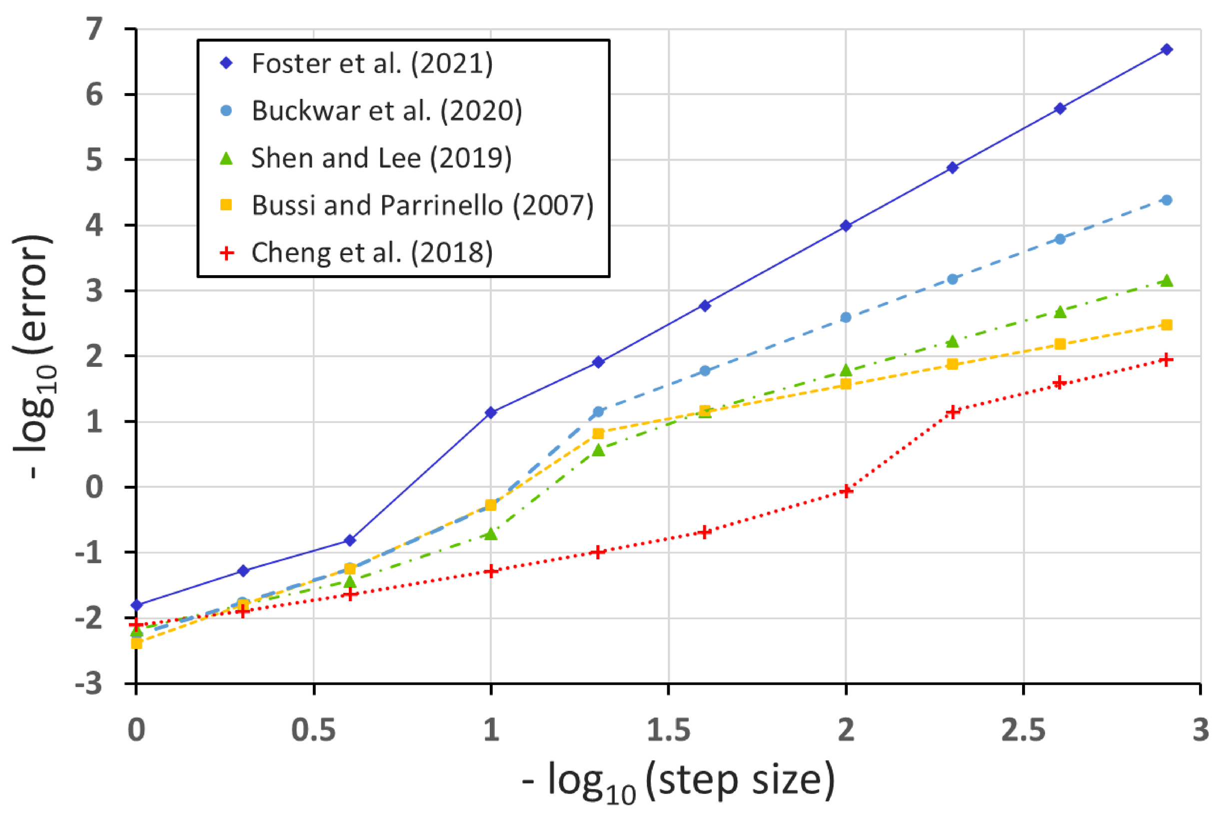
$$\frac{d^2x}{dt^2} = -\nabla f(x).$$

By adding friction and noise into Newton's second law, we obtain **Langevin dynamics:**

$$dx_t = v_t\,dt,$$
$$dv_t = -\gamma v_t\,dt - \nabla f(x_t)\,dt + \sqrt{2\gamma}\,dW_t.$$

As well as being a classical model in physics, Langevin dynamics has also been applied to sampling problems within machine learning.

As a result, developing numerical algorithms which can accurately simulate the Langevin dynamics remains an active area of research.




By leveraging our new-found knowledge of Brownian motion and (cubic) polynomials, we designed a novel algorithm for Langevin dynamics with state-of-the-art performance!

## References

[1] J. Foster, T. Lyons and H. Oberhauser, *An optimal polynomial approximation of Brownian motion*, SIAM Journal on Numerical Analysis, vol. 58, no. 3, pp.1393–1421, 2020.

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[3] J. Foster, T. Lyons and H. Oberhauser, *The shifted ODE method for underdamped Langevin MCMC*, http://arxiv.org/abs/2101.03446, 2021.